J Glob Optim (2010) 46:509–519
DOI 10.1007/s10898-009-9435-x

Nash equilibria of generalized games in normed spaces without upper semicontinuity

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Received: 5 May 2009 / Accepted: 7 May 2009 / Published online: 24 May 2009

Abstract The aim of this paper is to prove an existence theorem for the Nash equilibria of a noncooperative generalized game with infinite-dimensional strategy spaces. The main peculiarity of this result is the absence of upper semicontinuity assumptions on the constraint multifunctions. Our result is in the same spirit of the paper Cubiotti (J Game Theory 26: 267–273, 1997), where only the case of finite-dimensional strategy spaces was considered.

Keywords Noncooperative generalized game · Nash equilibria · Multifunctions · Upper semicontinuity

1 Introduction

Let $I = \{1, \ldots, N\}$ be a set of N players. Let each player $i \in I$ be endowed with a non-empty strategy space X_i, a utility function $p_i : X \to \mathbb{R}$ and a constraint multifunction $F_i : X_{-i} \to 2^{X_i}$, where we put

$$X := \prod_{i=1}^{N} X_i, \qquad X_{-i} := \prod_{\substack{j=1 \\ j \neq i}}^{N} X_j.$$

The family of triples $\{(X_i, p_i, F_i)\}_{i \in I}$ is called a (non-cooperative) *generalized game*, or an *abstract economy*. In the sequel, we shall assume that each set X_i is a nonempty subset of a real normed space E_i. The elements of the space X are called *multistrategies*. If vectors $x_i \in X_i$ $(i \in I)$ are given, we shall denote by x the multistrategy $x := (x_1, \ldots, x_N) \in X$. Conversely,

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if $x \in X$ is given, we shall denote by x_i the i-th subvector of x and by $x_{-i} \in X_{-i}$ the vector x without its i-th subvector x_i. If $x \in X$ and $v_i \in X_i$, we shall denote by $(x_{-i}, v_i) \in X$ the vector x with its i-th subvector x_i being replaced by v_i.

Let $F : X \to 2^X$ be the multifunction defined by setting, for each $x \in X$,

$$F(x) := \prod_{i=1}^{N} F_i(x_{-i}).$$

We recall that a vector $\hat{x} \in X$ is called a *generalized Nash equilibrium* for the game (see for instance [1–3,9,10]) if $x \in F(x)$ and for each $i \in I$ one has

$$p_i(\hat{x}_{-i}, v_i) - p_i(\hat{x}) \leq 0 \quad \text{for all} \quad v_i \in F_i(\hat{x}_{-i}).$$

As known, each multifunction F_i identifies the set of strategies for the player i which are allowed by the the other players' choice. That is, once the other players' strategy x_{-i} is given, the player i can choose his strategy only in the set $F_i(x_{-i})$, and not in the whole set X_i. Consequently, since the behaviour of the players is noncooperative, the aim of each player is to maximize his utility over the set $F_i(x_{-i})$. Thus, a multistrategy $\hat{x} \in X$ is a generalized Nash equilibrium if it is *feasible* (that is, $\hat{x} \in F(\hat{x})$) and it is a *no regret* strategy for each player. That is, none of the players can unilaterally improve his utility by choosing a different strategy, given the constraints imposed on him by the other players' action.

When for each $i \in I$ one has $F_i(x_{-i}) \equiv X_i$ (that is, the strategy space of each player is not affected by the other players' stategy), then the notion of generalized Nash equilibrium reduces to the classical notion of Nash equilibrium for a noncooperative N-person game in standard form.

As remarked in [19] (see also [1,2,6,8–11,14–16]), the standard existence theorems for the generalized Nash equilibria typically require both the upper and the lower semicontinuity of the multifunctions F_i, together with the convexity and the closedness of their values. They also typically assume convexity and compactness of the strategy spaces X_i, continuity of the functions p_i, and concavity (or quasiconcavity) of each p_i with respect to the i-th strategy x_i.

Recently, in the paper [3], some results were proved in the setting of finite-dimensional spaces E_i (both for bounded and unbounded strategy spaces X_i), where the typical upper semicontinuity condition on the constraint multifunctions F_i is not assumed. Indeed, such an assumption was replaced by the following more general condition: the feasible set

$$\{ x \in X : x \in F(x) \}$$

is closed. Moreover, some example where provided (to which we refer for a more detailed discussion) where such results apply. Here, we only recall that the upper semicontinuity and closed valuedness of each F_i, together with the compactness of each X_i, imply that the feasible set is closed (since the graph of F is closed—see Theorems 7.3.14 and 7.1.15 of [12]), while the converse is not necessarily true.

We remark that the finite-dimensionality assumption in the paper [3] was a key tool. The aim of this note is to extend the results of [3] to the setting of infinite-dimensional normed spaces E_i. This will be made by an approximation argument which makes use, in particular, of the finite-dimensional results of [3] and of some technical results concerning lower semicontinuous multifunctions. The technique we use is similar to the one originally developped in [5].

The following is our result:

Theorem 1.1 *Let $\{X_i, F_i, p_i\}_{i\in I}$ be an abstract economy. For each $i \in I$, let $K_i^{(1)}, K_i^{(2)} \subseteq X_i$ be nonempty compact sets, with $K_i^{(1)} \subseteq K_i^{(2)}$, $K_i^{(2)}$ convex, and $K_i^{(1)}$ finite-dimensional, such that the following assumptions are satisfied:*

(i) *X_i is a closed convex subset of the real normed space E_i;*
(ii) *p_i is continuous;*
(iii) *for each $x_{-i} \in X_{-i}$, the function $p_i(x_{-i}, \cdot)$ is concave on X_i;*
(iv) *the multifunction $F_i : X_{-i} \to 2^{X_i}$ is Hausdorff lower semicontinuous with closed convex values;*
(v) $\mathrm{int}_{\mathrm{aff}(X_i)}(F_i(x_{-i})) \neq \emptyset$ *for all $x_{-i} \in X_{-i}$;*
(vi) *$F_i(x_{-i}) \cap K_i^{(1)} \neq \emptyset$ for all $x_{-i} \in X_{-i}$.*
Moreover, assume that:
(vii) *the feasible set $\Delta := \{x \in X : x \in F(x)\}$ is compactly closed;*
(viii) *for each $x \in \Delta \setminus \left[\prod_{i=1}^N K_i^{(2)}\right]$, one has*

$$\max_{i\in I} \sup_{y_i \in F_i(x_{-i})\cap K_i^{(1)}} \left[p_i(x_{-i}, y_i) - p_i(x)\right] > 0.$$

Then there exists a generalized Nash equilibrium for the game.

The proof of Theorem 1.1 will be given in Sect. 3, while in Sect. 2 we shall fix some notations and recall some definitions and preliminary results which will be useful in the sequel. Finally, in Sect. 4, we shall discuss briefly about possible improvements of Theorem 1.1.

2 Preliminaries

For the basic facts about multifunctions, we refer to [12]. Here, for the reader's convenience, we only recall the following definitions. If S and Y are topological spaces and $\Phi : S \to 2^Y$ is a multifunction, we say that Φ is lower semicontinuous (resp., upper semicontinuous) at $x \in S$ if for each open set $A \subseteq Y$, with $\Phi(x) \cap A \neq \emptyset$ (resp., with $\Phi(x) \subseteq A$), the set $\Phi^-(A) := \{s \in S : \Phi(s) \cap A \neq \emptyset\}$ (resp., the set $\{s \in S : \Phi(s) \subseteq A\}$) is a neighborhood of x in S. We say that Φ is lower (resp., upper) semicontinuous in S if it is lower (resp., upper) semicontinuous at each point $x \in S$. The graph of Φ is the set $\{(s, y) \in S \times Y : y \in \Phi(s)\}$.

Let $(E, \|\cdot\|_E)$ be a real normed space. We say that a multifunction $\Phi : S \to 2^E$ is Hausdorff lower semicontinuous (resp., Hausdorff upper semicontinuous) at $x_0 \in S$ if for each $\sigma > 0$ there exists a neighborhood W of x_0 in S such that

$$\Phi(x_0) \subseteq \Phi(x) + B_\sigma \quad \text{for all} \quad x \in W$$
$$(\text{resp.,} \quad \Phi(x) \subseteq \Phi(x_0) + B_\sigma \quad \text{for all} \quad x \in W)\,,$$

where B_σ denote the open ball in E centered at the origin with radius σ. We say that Φ is Hausdorff lower (resp., Hausdorff upper) semicontinuous in S if it is Hausdorff lower (resp., Hausdorff upper) semicontinuous at each point $x \in S$. It is easy to check [12,18] that Hausdorff lower semicontinuity implies lower semicontinuity, and, conversely, upper semicontinuity implies Hausdorff upper semicontinuity. The converse implications are true if the values of Φ are nonempty and compact [12, Theorem 7.1.14].

Let $A \subseteq E$ be a nonempty set. We denote by $A)$ the affine hull of the set A. If $A \subseteq C \subseteq E$, we denote by $\mathrm{int}_C(A)$ the interior of A in C. Finally, we recall that the set $A \subseteq E$ is said to be compactly closed if its intersection with any compact subset of E is closed.

The following result will be a fundamental tool in the sequel.

Proposition 2.1 (Proposition 2.5 of [4]). *Let S be a topological space, $(E, \|\cdot\|_E)$ a real normed space, V an affine manifold of E, $\Phi : S \to 2^V$ an Hausdorff lower semicontinuous multifunction with nonempty closed convex values, and let $\overline{s} \in S$, $\overline{y} \in \mathrm{int}_V(\Phi(\overline{s}))$. Then, there exists a neighborhood U of $\overline{s}$ in S such that*

$$\overline{y} \in \mathrm{int}_V\left(\bigcap_{s\in U} \Phi(s)\right).$$

Let $\{(X_i, p_i, F_i)\}_{i\in I}$ be a generalized game, where each strategy space X_i is a nonempty subset of the real normed space $(E_i, \|\cdot\|_{E_i})$. In the sequel of the paper, in order to make notations simpler, we shall write $\prod_{i\in I}$ instead of the more correct symbol $\prod_{i=1}^N$. If $i \in I$, $x_i \in E_i$, and $r > 0$, we denote by $B_i(x_i, r)$ and $\overline{B}_i(x_i, r)$, respectively, the open ball and the closed ball in E_i centered at x_i with radius r. Moreover, if 0_{E_i} denotes the origin of the space E_i, we put

$$B_i(r) := B_i(0_{E_i}, r),$$
$$\overline{B}_i(r) := \overline{B}_i(0_{E_i}, r).$$

Finally, the product spaces

$$E := \prod_{i\in I} E_i\,, \qquad E_{-i} := \prod_{\substack{j\in I\\ j\neq i}} E_j$$

will be considered with the product topologies, generated by the norms

$$\|x\|_E = \max_{i\in I} \|x_i\|_{E_i}, \qquad \|x_{-i}\|_{E_{-i}} = \max_{\substack{j\in I\\ j\neq i}} \|x_j\|_{E_j}.$$

3 The proof of Theorem 1.1

For each $i \in I$, let $V_i := \mathrm{aff}(Xi)$, and let V_i^0 be the linear subspace of E_i corresponding to V_i (of course, V_i may not be closed in E_i). Following the notations of the previous sections, for each $i \in I$ we put

$$K_{-i}^{(2)} := \prod_{\substack{j\in I\\ j\neq i}} K_j^{(2)}, \qquad V_{-i} := \prod_{\substack{j\in I\\ j\neq i}} V_j\ .$$

We also put

$$K^{(2)} := \prod_{i\in I} K_i^{(2)}.$$

From now on, for the reader's convenience, we shall divide the proof into steps.

Step 1- Fix $i \in I$. For each $z_{-i} \in K_{-i}^{(2)}$, since $\mathrm{int}_{V_i}(F_i(z_{-i})) \neq \emptyset$ by assumption (v), choose any point

$$u_{(z_{-i})} \in \mathrm{int}_{V_i}\,(F_i(z_{-i})).$$

By Proposition 2.1, for each $z_{-i} \in K_{-i}^{(2)}$ there exists an open bounded neighborhood $W_{z_{-i}}$ of z_{-i} in E_{-i} such that

$$u_{(z_{-i})} \in \mathrm{int}_{V_i}\left(\bigcap_{v_{-i} \in W_{z_{-i}} \cap X_{-i}} F_i(v_{-i})\right). \tag{1}$$

Since $K_{-i}^{(2)}$ is compact, there exist vectors $z_{-i}^{(1)}, \ldots, z_{-i}^{(m_i)} \in K_{-i}^{(2)}$ such that

$$K_{-i}^{(2)} \subseteq \Omega_{(-i)} := \bigcup_{j=1}^{m_i}\left[W_{z_{-i}^{(j)}} \cap V_{-i}\right]. \tag{2}$$

Firstly we note that $\Omega_{(-i)}$ is open in V_{-i} and bounded (note that $\Omega_{(-i)}$ is not necessarily a product). Therefore, since the set $V_{-i} \setminus \Omega_{(-i)}$ is nonempty and closed in V_{-i}, and $K_{-i}^{(2)}$ is compact, by (2) we get

$$\xi(i) := \inf_{a_{-i} \in K_{-i}^{(2)}} \ \inf_{v_{-i} \in V_{-i} \setminus \Omega_{(-i)}} \| a_{-i} - v_{-i} \|_{E_{-i}} > 0. \tag{3}$$

Step 2- Once the number $\xi(i)$ is constructed for each $i \in I$, we put

$$\xi := \min_{i \in I} \xi(i).$$

For each $i \in I$, put

$$\Sigma_i := K_i^{(2)} + \left[\overline{B}_i\left(\frac{\xi}{2}\right) \cap V_i^0\right]. \tag{4}$$

It is easily seen that each Σ_i is convex and closed in V_i, and it is also bounded. Moreover, it follows by (3) that for each $i \in I$ one has also

$$\Sigma_{-i} := \prod_{\substack{j \in I \\ j \neq i}} \Sigma_j \subseteq \Omega_{(-i)}.$$

Step 3- For each $i \in I$, define $\mathcal{F}_i$ as the family of all finite-dimensional subspaces of E_i containing the set

$$K_i^{(1)} \cup \left\{u_{(z_{-i})}^{(1)}, \ldots, u_{(z_{-i})}^{(m_i)}\right\}.$$

Fix

$$S_1 \in \mathcal{F}_1, \quad \ldots, \quad S_N \in \mathcal{F}_N,$$

and let

$$S := S_1 \times \cdots \times S_N.$$

For each $i \in I$, define

$$X_i^S := \overline{X_i \cap \Sigma_i \cap S_i}.$$

Observe that, for each $i \in I$, one has

$$K_i^{(1)} \subseteq X_i \cap \Sigma_i \cap S_i \subseteq X_i^S \subseteq X_i \cap S_i.$$

In particular, $X_i^S \neq \emptyset$.

Step 4- For each $i \in I$, let

$$X_{-i}^S := \prod_{\substack{j \in I \\ j \neq i}} X_j^S,$$

and let $F_i^S : X_{-i}^S \to 2^{X_i^S}$ be the multifunction defined by setting, for each $x_{-i} \in X_{-i}^S$,

$$F_i^S(x_{-i}) := F_i(x_{-i}) \cap X_i^S = F_i(x_{-i}) \cap \overline{X_i \cap \Sigma_i \cap S_i}.$$

At this point, our aim is to apply Theorem 2.2 of [3] to the generalized game

$$\left\{X_i^S, F_i^S, p_i|_{X^S \times X^S}\right\}_{i \in I}, \tag{5}$$

where, as usual, we put $X^S := \prod_{i \in I} X_i$. To this aim, we observe the following facts.

(a) For each $i \in I$, the set X_i^S is a nonempty closed convex subset of S_i. Moreover, each X_i^S is bounded (and finite-dimensional), hence compact.

(b) For each $i \in I$, the multifunction $F_i^S : X_{-i}^S \to 2^{X_i^S}$ has nonempty convex values by (iv) and (vi) (since $K_i^{(1)} \subseteq X_i^S$).

(c) The feasible set of the game (5) is closed. Indeed, if for each $x \in X^S$ we put

$$F^S(x) := \prod_{i \in I} F_i^S(x_{-i}) = F(x) \cap X^S,$$

then the feasible set of the game (5) is the set

$$\Delta_S := \left\{x \in X^S : x \in F^S(x)\right\} = \Delta \cap X^S,$$

which is closed by assumption (vii).

(d) For each $i \in I$, the multifunction $F_i^S : X_{-i}^S \to 2^{X_i^S}$ is lower semicontinuous. To see this, fix $i \in I$ (recall that S is fixed).

Firstly, we prove that

$$\Sigma_i \cap S_i \cap \mathrm{int}_{V_i} F_i(x_{-i}) \neq \emptyset \quad \text{for all} \quad x_{-i} \in X_{-i}^S. \tag{6}$$

To prove (6), choose $x_{-i} \in X_{-i}^S$. For each $j \in I$, with $j \neq i$, let $x_j^* \in X_j \cap \Sigma_j \cap S_j$ such that $\|x_j - x_j^*\|_{E_j} \leq \xi/4$.

Hence, we can consider the point

$$x_{-i}^* \in \prod_{\substack{j \in I \\ j \neq i}} (X_j \cap S_j \cap \Sigma_j) \subseteq X_{-i}^S.$$

Note that

$$x_j - x_j^* \in V_j^0, \quad \text{for all} \quad j \in I, \quad \text{with} \quad j \neq i.$$

Since by (4) we have

$$x_j^* \in K_j^{(2)} + \left[\overline{B}_j\left(\frac{\xi}{2}\right) \cap V_j^0\right] \quad \text{for all} \quad j \in I, \quad j \neq i,$$

it follows that

$$x_j \in K_j^{(2)} + \left[\overline{B}_j\left(\frac{3\xi}{4}\right) \cap V_j^0\right] \quad \text{for all} \quad j \in I, \quad j \neq i,$$

hence by (3) we get

$$x_{-i} \in \Omega_{(-i)}.$$

Consequently, by (2), there exists $k \in \{1, \ldots, m_i\}$ such that

$$x_{-i} \in W_{z_{-i}^{(k)}} \cap V_{-i}.$$

By (1), we get in particular that $u_{z_{-i}^{(k)}} \in \mathrm{int}_{V_i}(F_i(x_{-i}))$, hence

$$u_{z_{-i}^{(k)}} \in S_i \cap \mathrm{int}_{V_i}(F_i(x_{-i})) \neq \emptyset.$$

By assumption (vi) we have $F_i(x_{-i}) \cap K_i^{(1)} \neq \emptyset$. Fix any point $v_i \in F_i(x_{-i}) \cap K_i^{(1)}$ (of course, $v_i \in S_i$ since $K_i^{(1)} \subseteq S_i$). The convexity of $F_i(x_{-i})$ implies that

$$v_i + t\left(u_{z_{-i}^{(k)}} - v_i\right) \in S_i \cap \mathrm{int}_{V_i}(F_i(x_{-i})) \quad \text{for all} \quad t \in]0, 1]. \tag{7}$$

On the other hand, since $K_i^{(1)} \subseteq K_i^{(2)}$, by (4) we have

$$v_i + \left[\overline{B}_i\left(\frac{\xi}{2}\right) \cap V_i^0\right] \subseteq \Sigma_i.$$

Consequently, we can find $\alpha \in]0, 1]$ such that

$$v_i + t(u_{z_{-i}^{(k)}} - v_i) \in \Sigma_i \quad \text{for all} \quad t \in]0, \alpha[. \tag{8}$$

In particular, by (7) and (8) we have

$$S_i \cap \Sigma_i \cap \mathrm{int}_{V_i}(F_i(x_{-i})) \neq \emptyset,$$

as desired. Thus, (6) is now proved. At this point we can prove that F_i^S is lower semicontinuous over X_{-i}^S. To this aim, let $\tilde{x}_{-i} \in X_{-i}^S$ and let A_i be an open set in V_i such that

$$F_i^S(\tilde{x}_{-i}) \cap A_i \neq \emptyset.$$

By (6) we have that

$$\Sigma_i \cap S_i \cap \mathrm{int}_{V_i}(F_i(\tilde{x}_{-i})) \neq \emptyset.$$

Consequently, there exists a point

$$w_i \in \Sigma_i \cap S_i \cap \mathrm{int}_{V_i}(F_i(\tilde{x}_{-i})) \subseteq F_i^S(\tilde{x}_{-i}).$$

Choose a point $\tilde{v}_i \in F_i^S(\tilde{x}_{-i}) \cap A_i$. Since the set $F_i(\tilde{x}_{-i})$ is convex, we have that

$$\tilde{v}_i + \lambda(w_i - \tilde{v}_i) \in X_i^S \cap \mathrm{int}_{V_i}(F_i(\tilde{x}_{-i})) \quad \text{for all} \quad \lambda \in]0, 1]. \tag{9}$$

On the other hand, since A_i is open in V_i, there exists $\mu > 0$ such that

$$\tilde{v}_i + \left[B_i(\mu) \cap V_i^0\right] \subseteq A_i. \tag{10}$$

Consequently, by (9) and (10), there exists $\delta \in]0, 1]$ such that

$$\tilde{v}_i + \delta(w_i - \tilde{v}_i) \in X_i^S \cap A_i \cap \mathrm{int}_{V_i}(F_i(\tilde{x}_{-i})). \tag{11}$$

By Proposition 2.1, there is a neighborhood $Z_{(-i)}$ of $\tilde{x}_{-i}$ in X_{-i} such that

$$\tilde{v}_i + \delta(w_i - \tilde{v}_i) \in \operatorname{int}_{V_i}\left(\bigcap_{u_{-i}\in Z_{(-i)}} F_i(u_{-i})\right). \tag{12}$$

By (11) and (12), we get

$$\tilde{v}_i + \delta(w_i - \tilde{v}_i) \in X_i^S \cap A_i \cap \operatorname{int}_{V_i}(F_i(u_{-i})) \quad \text{for all} \quad u_{-i} \in Z_{(-i)},$$

hence, in particular,

$$F_i(u_{-i}) \cap A_i \cap X_i^S \neq \emptyset \quad \text{for all} \quad u_{-i} \in Z_{(-i)},$$

and thus

$$F_i^S(u_{-i}) \cap A_i \neq \emptyset \quad \text{for all} \quad u_{-i} \in Z_{(-i)} \cap X_{-i}^S,$$

as desired.

Consequently, by Theorem 2.2 of [3], there exists $x^S \in X^S$ such that $x^S \in F^S(x^S)$ and for each $i \in I$ one has

$$p_i(x_{-i}^S, y_i) - p_i(x^S) \leq 0 \qquad \text{for all} \qquad y_i \in F_i^S(x_{-i}^S). \tag{13}$$

By (13) and assumption (viii), taking into account that $x^S \in \Delta$ and $K_i^{(1)} \subseteq X_i^S$ for all $i \in I$, we have that $x^S \in K^{(2)}$. We now prove that for each $i \in I$ one has

$$p_i(x_{-i}^S, y_i) - p_i(x^S) \leq 0 \qquad \text{for all} \qquad y_i \in F_i(x_{-i}^S) \cap S_i. \tag{14}$$

To this aim, fix $i \in I$ and $y_i \in F_i(x_{-i}^S) \cap S_i$. Since

$$\begin{aligned} &x_i^S \in K_i^{(2)} \subseteq X_i \subseteq V_i,\\ &y_i \in F_i(x_{-i}^S) \subseteq X_i \subseteq V_i,\\ &V_i - V_i \subseteq V_i^0, \end{aligned}$$

and X_i is convex, we have that

$$x_i^S + t(y_i - x_i^S) \in X_i \cap \left[K_i^{(2)} + \left(B_i\left(\frac{\xi}{2}\right) \cap V_i^0\right)\right] = X_i \cap \Sigma_i$$

for a sufficiently small $t \in]0,1[$. Hence, by the convexity of $F_i(x_{-i}^S)$ and by the definition of X_i^S, we have

$$x_i^S + t(y_i - x_i^S) \in X_i \cap \Sigma_i \cap S_i \cap F_i(x_{-i}^S) \subseteq X_i^S \cap F_i(x_{-i}^S) = F_i^S(x_{-i}^S).$$

By (13) and assumption (iii), we get

$$\begin{aligned} 0 &\geq p_i\left(x_{-i}^S, x_i^S + t\left(y_i - x_i^S\right)\right) - p_i(x^S)\\ &\geq (1-t)\, p_i(x^S) + t\, p_i\left(x_{-i}^S, y_i\right) - p_i(x^S)\\ &= t\left[p_i\left(x_{-i}^S, y_i\right) - p_i(x^S)\right], \end{aligned}$$

hence $p_i(x_{-i}^S, y_i) - p_i(x^S) \leq 0$, as desired.

Step 5- Resuming, we have proved that, if one fixes

$$S_1 \in \mathcal{F}_1, \quad \ldots, \quad S_N \in \mathcal{F}_N,$$

and puts

$$S = S_1 \times \cdots \times S_N,$$

then there exists a point $x^S \in K^{(2)} \cap S \cap \Delta$ such that for each $i \in I$ the relation (14) holds.

Now, let $\mathcal{F}$ be the family of all linear subspaces S of E of the type

$$S = S_1 \times \cdots \times S_N, \quad \text{with} \quad S_i \in \mathcal{F}_i.$$

Let us consider the net $\{x^S\}_{S\in\mathcal{F}}$, with $\mathcal{F}$ ordered by the ordinary set inclusion. The compactness of $K^{(2)}$ implies that the net $\{x^S\}_{S\in\mathcal{F}}$ has a cluster point $\hat{x} \in K^{(2)}$. Since by assumption (vii) the set $\Delta \cap K^{(2)}$ is closed, we get $\hat{x} \in F(\hat{x})$. We now claim that for each $i \in I$ one has

$$p_i(\hat{x}_{-i}, y_i) - p_i(\hat{x}) \leq 0 \quad \text{for all} \quad y_i \in \operatorname{int}_{V_i}(F_i(\hat{x}_{-i})). \tag{15}$$

Arguing by contradiction, assume that there exist $i \in I$ and $\tilde{y}_i \in \operatorname{int}_{V_i}(F_i(\hat{x}_{-i}))$ (which is nonempty by assumption (v)) such that

$$p_i(\hat{x}_{-i}, \tilde{y}_i) - p_i(\hat{x}) > 0. \tag{16}$$

By Proposition 2.1, there exist numbers $\sigma_j > 0$, with $j \in I$, $j \neq i$, such that

$$\tilde{y}_i \in \operatorname{int}_{V_i}\left(\bigcap_{x_{-i}\in(\prod_{j\in I, j\neq i} B_j(\hat{x}_j,\sigma_j))\cap X_{-i}} F_i(x_{-i})\right). \tag{17}$$

By (16) and assumption (ii), since the set

$$\{x \in X : p_i(x_{-i}, \tilde{y}_i) - p_i(x) > 0\}$$

is open in X, there exist numbers $\lambda_1, \ldots, \lambda_N > 0$, with $\lambda_j < \sigma_j$ for $j \neq i$, such that

$$X \cap \left[\prod_{j=1}^{N} B_j(\hat{x}_j, \lambda_j)\right] \subseteq \{x \in X : p_i(x_{-i}, \tilde{y}_i) - p_i(x) > 0\}. \tag{18}$$

By construction, there exists

$$\hat{S} = \hat{S}_1 \times \cdots \times \hat{S}_N \in \mathcal{F}$$

such that

$$\tilde{y}_i \in \hat{S}_i \quad \text{and} \quad x^{\hat{S}} \in \prod_{j=1}^{N} B_j(\hat{x}_j, \lambda_j).$$

By (17) we get

$$\tilde{y}_i \in \left[\operatorname{int}_{V_i} F_i\left(x_{-i}^{\hat{S}}\right)\right] \cap \hat{S}_i \subseteq F_i(x_{-i}^{\hat{S}}) \cap \hat{S}_i.$$

Consequently, (14) implies that

$$p_i(x_{-i}^{\hat{S}}, \tilde{y}_i) - p_i\left(x^{\hat{S}}\right) \leq 0. \tag{19}$$

On the other hand, (18) implies that

$$p_i(x^{\hat{S}}_{-i}, \tilde{y}_i) - p_i\left(x^{\hat{S}}\right) > 0,$$

which contradicts (19). Such a contradiction proves that (15) holds. At this point, by the convexity of each $F_i(\hat{x}_{-i})(i \in I)$ and the continuity of each p_i the conclusion follows at once. The proof is now complete. □

4 Final remarks

We now give some comments about the statement of Theorem 1.1 and possible improvements of it.

(a) Firstly, we observe that is not strictly necessary to assume the convexity of each set $K_i^{(2)}$. Indeed, in the proof of Theorem 1.1, each set $K_i^{(2)}$ can be replaced by its closed convex hull $\overline{co}\, K_i^{(2)}$, *provided that the last set is compact*. Consequently, the convexity assumption on each set $K_i^{(2)}$ in the statement of Theorem 1.1 can be replaced by the assumption that the set $\overline{co}\, K_i^{(2)}$ is compact. This happens, for instance, if the space E_i is a Banach space (see Theorem 6 at p.416 of [7]).

(b) Theorem 1.1 is a partial extension of Theorems 2.2 and 2.3 of [3] (note that when each space E_i is finite-dimensional one can take $K_i^{(1)} = K_i^{(2)}$). In particular, when the sets X_i are finite-dimensional and compact, and $F_i(x_{-i}) \equiv X_i$, Theorem 1.1 gives back the classical Nash existence theorem [2,14,15] (recall that each nonempty finite-dimensional convex set has nonempty interior in its affine hull).

(c) As regards possible improvements of Theorem 1.1, a first question could be: can the Hausdorff lower semicontinuity of the multifunctions F_i be replaced by the usual lower semicontinuity? In this connection, we point out that, in general, in infinite-dimensional setting, a lower semicontinuous multifunction has not the property described by Proposition 2.1, even if E is an Hilbert space (see Remark 1.1 of [13]). Moreover, one could ask if the finite-dimensional assumption on the sets $K_i^{(1)}$ can be dropped (and, consequently, one could take $K_i^{(1)} = K_i^{(2)}$). Our feeling is that it is not easy to give answers to these questions. In particular, as far as we know, the last problem has been unsuccessfully investigated by several mathematicians in the last years with respect to the variational inequality existence results proved in the papers [5,17].

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